Boussinesq System for Water Waves

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Thank you !!!

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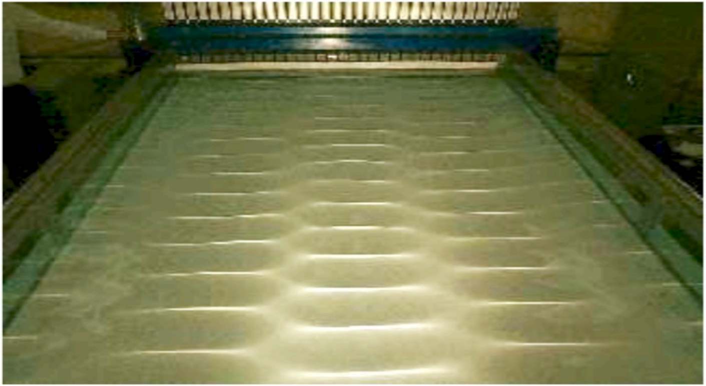
Water waves (Pic from D. Henderson and etc.)

- Unknowns: velocities $(u, v, w)(x, y, z, t)$,
- surface $\eta(x, y, t)$ (with 0 being the still water position),
- domain $\Omega(t) = (0, L) \times (0, H) \times (\tilde{h}, \eta(x, y, t))$,
- bottom topography $\tilde{h}(x, y, t)$, surface pressure $P(x, y, t)$.



A Boussinesq system for 3D waves (Bona, C., Saut (2002)

A Boussinesq system with moving bottom topography $\tilde{h}(x, y, t)$ and the surface pressure $P(x, y, t)$,

$$\begin{aligned} &\eta_t + \nabla \cdot \boldsymbol{v} + \nabla \cdot (h + \eta)\boldsymbol{v} - \frac{1}{6}\Delta\eta_t = F(h_{xxt}, h_{xtt}, \nabla P), \\ &\boldsymbol{v}_t + \nabla\eta + \frac{1}{2}\nabla|\boldsymbol{v}|^2 - \frac{1}{6}\Delta\boldsymbol{v}_t = G(h_{xxt}, h_{xtt}, \nabla P), \end{aligned} \qquad (\mathrm{BBM}^2)$$

where $h = \frac{\tilde{h}+h_0}{h_0}$ (flat means h=0) with h_0 = average water depth.

- The fluid is bounded by the bottom topography $\tilde{h}(x, y, t)$ and the free surface $\eta(x, y, t)$,
- $\eta(x, y, t)$ is a fundamental unknown of the problem,
- $\boldsymbol{v}(x, y, t)$ denotes the horizontal velocity at height $\sqrt{\frac{2}{3}}h_0$.

There are investigations on other Boussinesq systems, such as systems with KdV terms.

Justification (Bona, Colin, Lannes 2005)

- It is a first order approximation to Euler equations . Meaning: for any initial value $(\eta_0, u_0) \in H^\sigma(\mathbb{R})^2$ with $\sigma \geq s \geq 0$ large enough, there exists a unique solution $(\eta_{euler}, u_{euler})$ of Euler equations, such that

$$\|u - u_{euler}\|_{L^\infty(0,t;H^s)} + \|\eta - \eta_{euler}\|_{L^\infty(0,t;H^s)} = O(\epsilon_1^2 t, \epsilon_2^2 t, \epsilon_1 \epsilon_2 t)$$

  for $0 \leq t \leq O(\epsilon_1^{-1}, \epsilon_2^{-1})$.
- Other relevant works (Craig 1985,Schneider and Wayne 2000, Bona, Prichard, Scott 1981, Alazman, Albert, Bona, Chen, Wu (2003)...
- More justification needed, especially by comparing with data from the field and laboratory experiments.

Advantages of Boussinesq system

From practical side: it is

- Physically relevant, especially with BVP;
- Easy to analyze, to simulate, and to incorporate into a complex system when compared with Euler equation or Navier Stokes or ...,
- More accurate when compared with the Linear equation (lol).
- It has many desired and helpful properties, such as the existence of solitary waves, conservation of mass, $\cdots$.

Example, taking $h = P = 0$ and considering one-space dimension yields.

$$\eta_t + u_x + (\eta u)_x - \frac{1}{6}\eta_{xxt} = 0,$$
$$u_t + \eta_x + uu_x - \frac{1}{6}u_{xxt} = 0.$$

Properties of the system (Bona, C., Saut (2002))

- (Bona, C., Saut 2003) The linearized system is globally well posed in L^p and in W_p^k for $1 \le p \le \infty$ and $k = 0, 1, 2, \cdots$.
- It has the invariant functionals

$$H(\eta, u) = \frac{1}{2} \int_{-\infty}^{\infty} u^2(1+\eta) + \eta^2 dx,$$

$$I(\eta, u) = \int_{-\infty}^{\infty} u\eta + \frac{1}{6}\eta_x u_x dx,$$

$$I_u = \int_{-\infty}^{\infty} u\, dx \quad \text{and} \quad I_\eta = \int_{-\infty}^{\infty} \eta\, dx.$$

- there is a Hamiltonian structure based on H and I, namely

$$\partial_t \nabla_{(\eta,u)} I(\eta, u) + \partial_x \nabla_{(\eta,u)} H(\eta, u) \; = \; 0.$$

Wellposedness of the Boussinesq system

- These invariants are useful, but none of these invariants are composed only of positive terms, so they do not on their own provide the *a priori* information one needs to conclude global existence of solutions of initial-value problems, except when $1 + \eta \geq \alpha > 0$.
- Global well-posedness in time for the nonlinear problem is proved in [BCS] and [AABCW] under the condition that if there is an $\alpha > 0$ such that the solution satisfies

  $$1 + \eta(x, t) \geq \alpha \quad \text{for all} \quad t \geq 0.$$

  Note: Not a perfect result because it is based on an assumption about the solution η which is not known. In physical terms, the condition simply means that the bed does not run dry at any time, or what is the same, the free surface never touches the bottom.
- For some initial data, solution exists for all time. Examples, the exact nontrivial solution. For others?

Numerical study on global existence (Bona and C(2016))

We tested initial data of the form

$$\eta(x,0) = a\,e^{-x^2}, \quad u(x,0) = b\,xe^{-x^2}$$

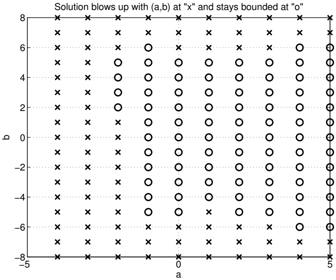


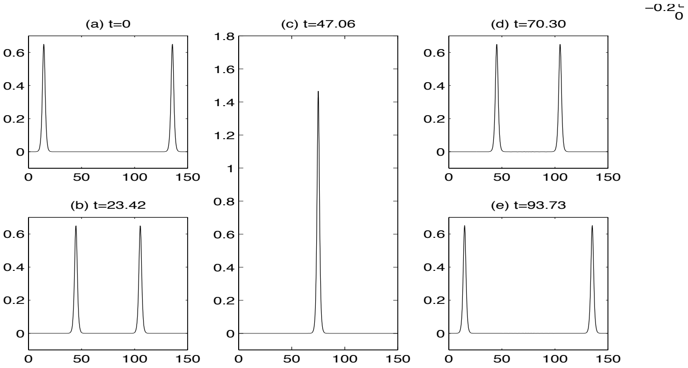
Figure: Blowup map for values of the parameters *a* and *b*. *x* blows up and the circles are values where the solution appears global.

Conjectures on global solutions in time

- *a* close to 0, small amplitude; $a > -1$, no dry up ;
- When *a* and *b* are small (and $a > -1$), namely in the physically relevant modeling range, it seems the global solution exists;
- $\eta(x,0) + 1 \geq \alpha > 0$ does not guarantee $\eta(x,t) + 1$ positive for all t;
- theoretical proof OPEN
- it seems to have a local similarity structure near the blowup point. Theoretical proof is also open

Head-on collision of solitary waves

- the solitary waves are generated numerically;
- there is a small phase shift after the collision;
- the amplitude at the middle of the interaction is larger than the combination of two incoming waves.



The 2-D Wave tank at PSU (Henderson and Hammack)

The Boussinesq system has these double periodic solutions.

Wave patterns: linear plane waves

- the 2D patterns are the oblique interaction of two plane waves;
- parameters involved in describing a plane wave:
  - traveling direction: $\mathbf{c} = c_0(1, 0)$, so the direction is in the y-direction;
  - the angle of the plane wave and the wave length of the plane wave $\mathbf{k}_1 = l_1(1, \tau_1)$;
- to search for this plane wave means to find solutions in the form of

$$\eta(\mathbf{x}) = \eta_{\mathbf{k_1}} e^{i\mathbf{k_1}\cdot(\mathbf{x}-\mathbf{c}t)}, \quad \mathbf{v}(\mathbf{x}) = \mathbf{v}_{\mathbf{k_1}} e^{i\mathbf{k_1}\cdot(\mathbf{x}-\mathbf{c}t)};$$

- substitute this ansatz into the linear part of the equations

$$\begin{aligned} \eta_t + \nabla \cdot \boldsymbol{v} + \nabla \cdot \eta \boldsymbol{v} - \frac{1}{6}\Delta\eta_t &= 0, \\ \boldsymbol{v}_t + \nabla\eta + \frac{1}{2}\nabla|\boldsymbol{v}|^2 - \frac{1}{6}\Delta\boldsymbol{v}_t &= 0, \end{aligned} \qquad (\mathrm{BBM}^2)$$

Wave patterns: linear plane waves

- $\mathbf{k}_1$, $\mathbf{c}$, $\eta_{\mathbf{k}_1}$ and $\mathbf{v}_{\mathbf{k}_1}$ satisfy

$$
\begin{aligned}
-(1+\frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\eta_{\mathbf{k}} + \mathbf{k}\cdot\mathbf{v}_{\mathbf{k}} &= 0,\\
\mathbf{k}\eta_{\mathbf{k}} - (1+\frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\mathbf{v}_{\mathbf{k}} &= 0.
\end{aligned}
$$

- For the nontrivial ($(\eta_{\mathbf{k}}, \mathbf{v}_{\mathbf{k}}) \neq \mathbf{0}$) solutions (plane waves) to exist, $\mathbf{k}_1$ and $\mathbf{c}$ are such that the determinant is zero, i.e. satisfy dispersion relation

$$
\Delta(\mathbf{k},\mathbf{c}) = (1+\frac{1}{6}|\mathbf{k}|^2)^2(\mathbf{c}\cdot\mathbf{k})^2 - |\mathbf{k}|^2 = 0. \qquad \text{(Det)}
$$

- Similarly, for the other plane wave,

$$
\eta(\mathbf{x}) = \eta_{\mathbf{k}_2} e^{i\mathbf{k}_2\cdot(\mathbf{x}-\mathbf{c}t)}, \quad \mathbf{v}(\mathbf{x}) = \mathbf{v}_{\mathbf{k}_2} e^{i\mathbf{k}_2\cdot(\mathbf{x}-\mathbf{c}t)},
$$

where $\mathbf{k}_2 = l_2(1, -\tau_2)$, $\mathbf{k}_2$ and $\mathbf{c}$ have to satisfy (Det).

Sketch of the linear study on wave patterns

Assume $\mathbf{k}_1$ and $\mathbf{k}_2$ are the solutions to (Det), then we have wave patterns with parameters consist of 5 parameters c_0, l_1, l_2, τ_1 and τ_2

- 3 parameter families of patterns because (Det) has to be satisfied by $(\mathbf{k}_1, \mathbf{c})$ and $(\mathbf{k}_2, \mathbf{c})$, and
- amplitude $\eta_{\mathbf{k}_1}$, $\eta_{\mathbf{k}_2}$, $\mathbf{v}_{\mathbf{k}_2}$, $\mathbf{v}_{\mathbf{k}_1}$;
- symmetric lattice: $\tau_1 = \tau_2$, $l_1 = l_2$;
- symmetric pattern: symmetric lattice plus

$$\eta_{\mathbf{k}_1} = \eta_{\mathbf{k}_2}, \quad \mathbf{v}_{\mathbf{k}_2} = \mathbf{v}_{\mathbf{k}_1}.$$

For symmetric patterns, two parameters for the lattice and half of the number of parameters for the amplitudes.

Results of the nonlinear study on wave patterns (C. and Iooss (2006))

Idea:

- add the nonlinear term in, using a perturbation approach (Lyapunov Schmidt);
- invert the linear operator around the kernel and find the bound for the pseudo-inverse;
- perturbation parameter: w in $\mathbf{c} = c_0(1, w)$ and amplitudes of the plane waves;

Results:

- existence of symmetric patterns with almost all parameters;
- existence of asymmetric patterns with symmetric lattice with almost all parameters;
- existence of asymmetric patterns with asymmetric lattice for a large set of parameters (small devisor problem occurs).

Theorem on symmetric wave patterns.

For symmetric lattice (2 parameters) with symmetric pattern, we have

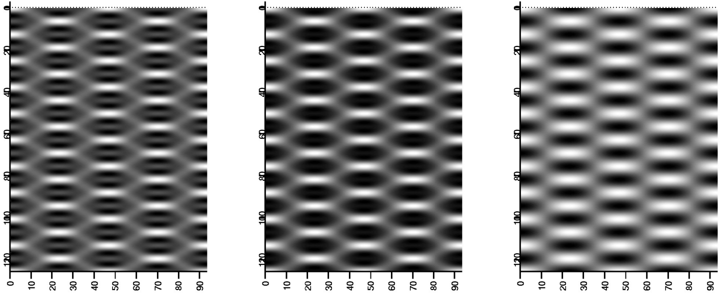
Theorem

(Chen and Iooss 2006) For almost every (k,τ), *k represents the wave number in y direction and τ represents the ratio between the periods in y and x directions, One example of the form of the free surface, even in y, is given by*

$$\eta = \varepsilon \cos ky \cos k\tau x - \frac{\varepsilon^2}{2(1+\tau^2)}\{\frac{1-\tau^2}{4} \cos 2k\tau x$$
$$+ c_1 \cos 2ky + d_1 \cos 2ky \cos 2k\tau x\} + O(\varepsilon^3).$$

Wave patterns with change of water depth (moving down)

From left to right with $h_0 = 0.5, 1, 8$, it is observed that the wave patterns are changing from "hexagonal" shape to "rectangular" shape.



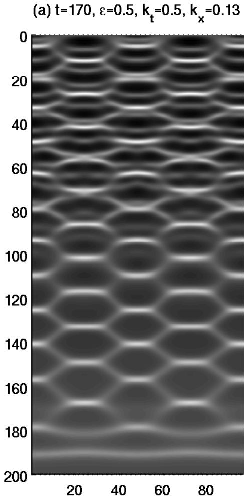
Standing wave patterns (C., Iooss and C. Shenghao Li (2017)

With similar techenique, we can prove the

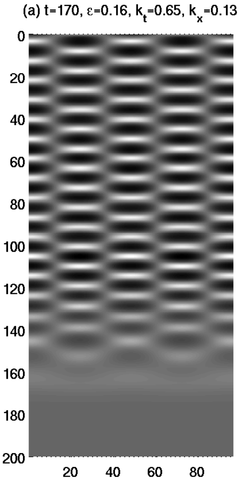
- existence of standing waves in one space dimension (C. and Iooss (2005)) and
- standing wave patterns in two space dimension (C. and Shenghao Li (2017)).
- A movie on standing wave patterns. Chrome and open file wave2.gif, bookmarked It is also available at `https://www.youtube.com/watch?v=7n3y55yIzxk`

Numerical and experimental

Numerical solution with nonzero boundary data at $y = 0$ only.
Zero initial data.

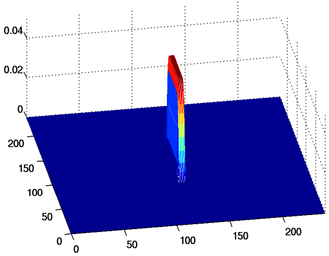


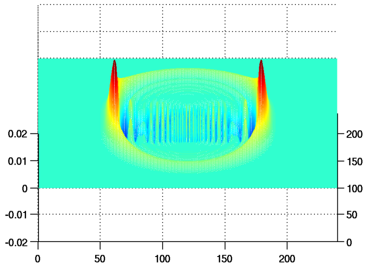
Numerical and experimental



Wave profile started from a rectangular source

Surface profile at t=0 and t=60 ($\eta(x, y, 0)$ and $\eta(x, y, 60)$) with aspect ratio $\sigma = 10$.





Boussinesq system with local dissipation and decay rate (C. and Goubet (2009)

$$\eta_t + u_x + (u\eta)_x - \eta_{xxt} = \eta_{xx},$$
$$u_t + \eta_x + uu_x + -u_{xxt} = u_{xx}.$$

Theorem

For initial data in $(H^1(\mathbb{R}) \cap L^1(\mathbb{R}))^2$ and small enough in $L^1 \cap L^2$, as $t \to \infty$,

$$\|\eta(t)\|^2_{L^2_x} + \|u(t)\|^2_{L^2_x} \leq O(t^{-1/2});$$

$$\|(\eta, u)\|_{L^\infty_x} \leq O(t^{-1/2}).$$

Single equation with white noise dispersion (C., Goubet and Mammeri (2016))

$$du - du_{xx} + u_x \circ dW + u^p u_x dt = 0,$$

in the Stratonovich formulation

- $W(t)$ is a standard real valued Brownian motion;
- the corresponding deterministic case: $u_t - u_{txx} + u_x + u^p u_x = 0$,
- assume $p > 8$ and the initial data is small in $L^1_x \cap H^4_x$;
- the solutions decays to zero at $O(t^{-\frac{1}{6}})$, instead of $O(t^{-\frac{1}{3}})$ as in deterministic case;

Other works, not a complete list at all

- Existence of solitary and multi-pulsed solutions (C. 2000)) and stabilities (Bona and C(1998));
- Existence of solitary wave solutions (C., Nguyen and Sun (2011));
- Existence of cnoidal wave solutions (Chen, C, Nguyen(2007)), explicit and topological index theory;
- Stability of solitary waves of elevation and depresion, stability of cnoidal waves (CCDLN (2010))
- Existence of solitary wave and their stability (C. Nguyen and Sun), for systems with large surface tension;
- non-even bottom topography, surface pressure change in (x, t) (C.);
- surface tension effects ();
- viscosity, add hoc local operator, nonlocal operators (C., Dumont, Goubet 2012) wellposedness and decay rate;
- model with stochastic dispersion (C., Goubet, Mammeri 2016) wellposedness and decay rate for equation

URL and Thank you!

References for the talk are available at

http://www.math.purdue.edu/~chen/pub.html

Thank you for your attention!